2.6 - Logical Equivalence

The crux of showing that two logical statements are 'logically equivalent' is to show that they have the same truth value in every situation, i.e., they have identical truth tables.

Recall that we actually showed that one of De Morgan's laws was true a few lectures back.

Let's examine here a few more examples:

Ex: Show that the following statements are logically equivalent:

ⓐ $P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$

ⓑ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

ⓒ $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

Sol:

ⓐ

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$(\sim Q) \Rightarrow (\sim P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

same

35

(b)

P	Q	R	Q∨R	P∧(Q∨R)	P∧Q	P∧R	(P∧Q)∨(P∧R)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

same ✓

(c)

P	Q	R	Q∨R	P∨(Q∨R)	P∨Q	(P∨Q)∨R
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	F	T
F	F	F	F	F	F	F

same

Ex: Are the following statements logically equivalent?

ⓐ $\sim(P \Rightarrow Q)$ and $\sim Q \Rightarrow \sim P$

ⓑ $P \wedge Q$ and $\sim(\sim P \vee \sim Q)$

Sol: ⓐ

P	Q	$P \Rightarrow Q$	$\sim(P \Rightarrow Q)$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	T	F	F	F	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	F	T	T	T

different

Not equivalent.

ⓑ

P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$	$\sim(\sim P \vee \sim Q)$

Instead of filling this out, we can use properties of logic:

$$\sim(\sim P \vee \sim Q) \overset{*}{=} \sim[\sim(P \wedge Q)] \overset{*}{=} P \wedge Q$$

$(\sim(P \wedge Q) = \sim P \vee \sim Q$ by DeMorgan's law$)$

$(\sim(\sim P) = P$ by reflexivity$)$

Logical Equivalences

1. Contrapositive Law $P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$

2. DeMorgan's Laws $\begin{cases} \sim(P \vee Q) = \sim P \wedge \sim Q \\ \sim(P \wedge Q) = \sim P \vee \sim Q \end{cases}$

3. Commutative Laws $\begin{cases} P \wedge Q = Q \wedge P \\ P \vee Q = Q \vee P \end{cases}$

4. Distributive Laws $\begin{cases} P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R) \end{cases}$

5. Associative Laws $\begin{cases} P \wedge (Q \wedge R) = (P \wedge Q) \wedge R \\ P \vee (Q \vee R) = P \vee (Q \vee R) \end{cases}$

2.7: Quantifiers

We studied the operations $\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$ so far, but how do we construct a stament like "for every $n \in \mathbb{Z}$, $2n$ is even."? How about "given integers a and b with $b > 0$, there exist integers q and r such that $a = qb + r$ with $0 \leq r < b$."

We have two quantifiers: $\exists$ and $\forall$

$\exists$: there exists / there is a (existential quant.)
$\forall$: for all / for every (universal quant.)

So, "for every $n \in \mathbb{Z}$, $2n$ is even" translates to:

$$\forall n \in \mathbb{Z},\ 2n \text{ is even}$$

or

$$\forall n \in \mathbb{Z},\ E(2n)$$

where $E(x)$ is the open sentence "x is even"

For a statement like "there exists $x \in \mathbb{R}$ such that $\ln x = 1$" we would have:

$$\exists x \in \mathbb{R},\ \ln x = 1$$

As for the division algorithm, it becomes

$$\forall a \in \mathbb{Z}, \forall b \in \mathbb{N}, \exists q, r \in \mathbb{Z},\ a = qb + r,\ 0 \leq r < b$$

Ex: Write the following statements in symbols:

(a) For every $y \in Y$ there is an $x \in X$ such that $f(x) = y$

(b) The empty set is a subset of every set.

Sol: (a) $\forall y \in Y, \exists x \in X,\ f(x) = y$

(b) $\forall X,\ \emptyset \subset X$